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Lieb-Thirring Inequalities for Fourth-Order Operators in Low Dimensions

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Abstract

This paper considers Lieb-Thirring inequalities for higher order differential operators. A result for general fourth-order operators on the half-line is developed, and the trace inequality

$$\operatorname{tr}\left((-\Delta)^2 - C_{d,2}^{\mathrm{HR}}\frac{1}{|x|^4} - V(x)\right)_-^\gamma \leq C_\gamma \int_{\mathbb{R}^d} V(x)_+^{\gamma+\frac{d}{4}}\,dx, \quad \gamma \geq 1-\frac{d}{4},$$

where $C_{d,2}^{\mathrm{HR}}$ is the sharp constant in the Hardy-Rellich inequality and where $C_\gamma > 0$ is independent of V, is proved for dimensions $d = 1, 3$. As a corollary of this inequality a Sobolev-type inequality is obtained.

1 Introduction

This paper concerns Lieb-Thirring inequalities with critical exponents. Well-known results in this area are the Lieb-Thirring inequalities

$$\operatorname{tr}\left((-\Delta)^l - V\right)_-^\gamma \leq C \int_{\mathbb{R}^d} V(x)^{\gamma+\frac{d}{2l}}\,dx, \quad \gamma \geq 1-\frac{d}{2l},$$

in the space $L^2\left(\mathbb{R}^d\right)$, where $l > d/2$, as discussed in [Wei96] and [NW96].

Recent papers such as [EF06] and [FLS08] combine Lieb-Thirring inequalities with the sharp Hardy-Rellich inequalities of the type

$$\int_{\mathbb{R}^d} \left|\nabla^l u(x)\right|^2 dx \geq C_{d,l}^{\mathrm{HR}} \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^{2l}}\,dx, \tag{1.1}$$

where $l < d/2$, as discussed in [Yaf99]. This case, however, does not admit a critical exponent. The inequalities thus obtained are of the type

$$\operatorname{tr}\left((-\Delta)^l - C_{d,l}^{\mathrm{HR}}\frac{1}{|x|^{2l}} - V(x)\right)_-^\gamma \leq C \int_{\mathbb{R}^d} V(x)_+^{\gamma+\frac{d}{2l}}\,dx, \quad \gamma > 0.$$

For inequalities with critical exponent, we again turn to the sharp Hardy-Rellich inequality (1.1), but this time we assume that $l > d/2$ and $l - d/2 \notin \mathbb{Z}$. In this case the inequality is valid for $u \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$. In [EF08], the following inequality is obtained for the case $l = d = 1$:

$$\operatorname{tr}\left(-\frac{d^2}{dx^2} - C_{1,1}^{\mathrm{HR}}\frac{1}{x^2} - V(x)\right)_-^\gamma \leq C\int_0^\infty V(x)_+^{\gamma+\frac{1}{2}}\,dx, \quad \gamma \geq \frac{1}{2},$$

where the operator on the left-hand-side is taken with Dirichlet boundary conditions at 0. In the present paper, we develop these techniques further to prove the critical exponent inequality

$$\operatorname{tr}\left((-\Delta)^2 - C_{d,2}^{\mathrm{HR}}\frac{1}{|x|^4} - V(x)\right)_-^\gamma \leq C_\gamma\int_{\mathbb{R}^d} V(x)_+^{\gamma+\frac{d}{4}}\,dx, \quad \gamma \geq 1 - \frac{d}{4}, \tag{1.2}$$

for the fourth-order cases $l = 2$ and $d = 1, 3$, where the constant $C_\gamma > 0$ is independent of V. Again, the operator in question is considered with Dirichlet conditions at 0.

In fact, we prove such an inequality for a general fourth-order operator on the half-line, from which the results for the bi-laplacian with Hardy weight in dimensions $d = 1, 3$ follow. This way we actually get a more general result than (1.2) in the case $d = 1$, by introducing a weight in the integral on the right-hand side. Such weighted inequalities exist for any $\gamma > 0$, and the weight can be chosen such that γ is still the critical exponent.

The methods for proving this general result origin in [Wei96], [NW96] and [EF08]. In this paper no Sturm-Liouville or Green's function theory is needed. One interesting technical result is the Sobolev-type inequality of Lemma 3.2.

It is worth noting that the proofs employed here can be extended to higher order $l \geq 3$ and dimensions d such that $l > d/2$ and $l - d/2 \notin \mathbb{Z}$, even if this would be somewhat tedious.

Finally, an immediate consequence of inequality (1.2) is a Sobolev-type inequality that estimates the L^p-norm of a function $u \in C_0^\infty(\mathbb{R}\setminus\{0\})$, for $1 < p \leq \infty$.

2 Main Results

We prove trace inequalities in dimensions $d = 1, 3$ for the fourth-order operator

$$H = H_0 - V, \quad \text{where} \quad H_0 = (-\Delta)^2 - C_{d,2}^{\mathrm{HR}}\frac{1}{|x|^4}.$$

Theorem 2.1. *Let $0 \leq \nu < 3$ and $\gamma \geq (3-\nu)/4$. Then, for any non-negative V such that $V(x)^{\gamma+(1+\nu)/4}x^\nu$ is integrable on $(0,\infty)$, the form*

$$u \mapsto \int_0^\infty \left(|u''(x)|^2 - C_{1,2}^{\mathrm{HR}}\frac{|u(x)|^2}{x^4} - V(x)|u(x)|^2\right)dx,$$

is lower semi-bounded on $C_0^\infty(0,\infty)$. Let $H_0 - V$ be the self-adjoint operator corresponding to the closure of this form. Then the negative spectrum of $H_0 - V$

is discrete, and there is a constant $C = C(\nu, \gamma) > 0$, independent of V, such that

$$\operatorname{tr}(H_0 - V)_-^\gamma \le C \int_0^\infty V(x)^{\gamma + \frac{1+\nu}{4}} x^\nu \, dx.$$

The weight in the integral on the right-hand side is important for two different reasons. First of all, it allows us to consider arbitrarily small $\gamma > 0$. Second, it allows us to pass to higher dimensions, as seen in Lemma 4.5 which is an important part of the proof of the following theorem:

Theorem 2.2. *Let $\gamma \ge 1/4$. For any non-negative $V \in L^{\gamma+3/4}\left(\mathbb{R}^3\right)$, the form*

$$u \mapsto \int_0^\infty \left(|\Delta u(x)|^2 - C_{3,2}^{\mathrm{HR}} \frac{|u(x)|^2}{|x|^4} - V(x)|u(x)|^2 \right) dx,$$

is lower semi-bounded on $C_0^\infty\left(\mathbb{R}^3 \backslash \{0\}\right)$. Let $H_0 - V$ be the self-adjoint operator corresponding to the closure of this form. Then the negative spectrum of $H_0 - V$ is discrete, and there is a constant $C = C(\gamma) > 0$, independent of V, such that

$$\operatorname{tr}(H_0 - V)_-^\gamma \le C \int_{\mathbb{R}^3} V(x)^{\gamma + \frac{3}{4}} \, dx.$$

In fact, these results follow from a result for a class of general fourth-order Schrödinger operators on the half-line. The proof of this result captures all the essential ideas in this paper, and can be used to prove similar results for other fourth-order operators than the bi-laplacian with a Hardy term.

Theorem 2.3. *Let $\alpha \ge 0$, $\beta \ge 0$, $0 \le \nu < 3$, $\nu \le 2\beta$ and $\gamma \ge (3 - \nu)/4$. Then, for any non-negative V such that $V(x)^{\gamma + (1+\nu)/4} x^\nu$ is integrable on $(0, \infty)$, the form*

$$u \mapsto \int_0^\infty \left(\left| \frac{d}{dx} \left(\frac{1}{x^\alpha} \frac{d}{dx} \left(\frac{u(x)}{x^\beta} \right) \right) \right|^2 x^{2(\alpha+\beta)} - V(x)|u(x)|^2 \right) dx,$$

is lower semi-bounded on $C_0^\infty(0, \infty)$. Let $H_0 - V$ be the self-adjoint operator corresponding to the closure of this form. Then the negative spectrum of $H_0 - V$ is discrete, and there is a constant $C = C(\alpha, \beta, \nu, \gamma) > 0$, independent of V, such that

$$\operatorname{tr}(H_0 - V)_-^\gamma \le C \int_0^\infty V(x)^{\gamma + \frac{1+\nu}{4}} x^\nu \, dx.$$

The proofs of these three theorems are postponed until Section 5

An immediate consequence of these theorems is the following Sobolev-type inequality:

Corollary 2.4. *Let $d = 1, 3$ and $1 < p \le \infty$. Then there are constants $D_1, D_2 > 0$ such that*

$$\left(\int_0^\infty |u|^{2p} dx \right)^{\frac{1}{p}} \le \int_0^\infty \left(|u''(x)|^2 - D_1 \frac{|u(x)|^2}{x^4} + D_2 |u(x)|^2 \right) dx, \quad u \in C_0^\infty\left(\mathbb{R}^d \backslash \{0\}\right).$$

Proof. We prove only the case $d = 1$, as the other case is similar. Let $q \geq 1$ be such that $p^{-1} + q^{-1} = 1$. Setting $\nu = 0$ and $\gamma = q - 1/4$ in Theorem 2.1 and letting

$$E(V) = \inf \sigma\left(H_0 - V\right)$$

we obtain that

$$E(V)^\gamma \geq -C\|V\|_q^q \, dx,$$

for any non-negative $V \in L^q\left(0, \infty\right)$, where $C > 0$ is independent of V. It follows that

$$H_0 - V + C^{1/\gamma}\|V\|_q^{q/\gamma} \geq 0,$$

and hence

$$\int_0^\infty \left(|u''(x)|^2 - C_{1,2}^{\mathrm{HR}}\frac{|u(x)|^2}{x^4} - V(x)|u(x)|^2 + C^{1/\gamma}\|V\|_q^{q/\gamma}|u(x)|^2\right) dx \geq 0, \tag{2.1}$$

for any $u \in C_0^\infty(0, \infty)$ and any non-negative $V \in L^q\left(0, \infty\right)$. Fix $u \in C_0^\infty(0, \infty)$ and consider the linear functional $L_u : L^q\left(0, \infty\right) \to \mathbb{C}$ given by

$$L_u V = \int_0^\infty V|u|^2 \, dx.$$

Choose $V \in L^p\left(0, \infty\right)$ with $\|V\|_q = 1$ and write $V = V_R + iV_I$, where V_R and V_I are real-valued. Furthermore, let V_R^+ and V_R^- be the positive and negative parts of V^R, respectively. Define V_I^+ and V_I^- similarly. Note that $\|V_R^+\|_q \leq \|V\|_q = 1$ and hence by (2.1),

$$L_u V_R^+ \leq \int_0^\infty \left(|u''(x)|^2 - C_{1,2}^{\mathrm{HR}}\frac{|u(x)|^2}{x^4} + C^{1/\gamma}|u(x)|^2\right) dx,$$

and similarly for V_-^R, V_+^I and V_-^I. Hence L_u is a bounded functional with

$$\|L_u\| \leq \int_0^\infty \left(|u''(x)|^2 - C_{1,2}^{\mathrm{HR}}\frac{|u(x)|^2}{x^4} + C^{1/\gamma}|u(x)|^2\right) dx, \tag{2.2}$$

The Riesz representation theorem give us that $|u|^2 \in L^p\left(0, \infty\right)$ with

$$\left\||u|^2\right\|_p = \|L_u\|$$

and since $u \in C_0^\infty(0, \infty)$ was arbitrary, the result follows from (2.2). ☐

3 An Auxiliary Operator on a Finite Interval

In this section, some auxiliary results for a certain operator on a finite subinterval of $(0, \infty)$ will be proved. These results are the key ingredients in the proof of Theorem 2.3. Throughout this section, the constants α and β will be fixed and satisfy

$$\alpha \geq 0 \quad \text{and} \quad 0 \leq \beta < \frac{3}{2} + \alpha.$$

For $b > 0$, define the closed quadratic form

$$h_b[u] = \int_b^{b+1} \left| \left(\frac{1}{x^\alpha} \left(\frac{u(x)}{x^\beta} \right)' \right)' \right|^2 x^{2(\alpha+\beta)} \, dx,$$

with domain $D[h_b] = H^2(b, b+1)$. As usual, there is a canonically defined sesqui-linear form, denoted by $h_b[\cdot,\cdot]$ from which h_b arises, but we will make no distinction between the quadratic and sesqui-linear forms. Let H_b denote the self-adjoint operator in $L^2(b, b+1)$ associated with h_b. We will henceforth fix the functions

$$f_1(x) = x^\beta \quad \text{and} \quad f_2(x) = x^{\alpha+\beta+1}.$$

and note that $f_1, f_2 \in D[h_b]$, as well as $h_b[f_1, v] = h_b[f_2, v] = 0$ for any function $v \in D[h_b]$. Therefore, $f_1, f_2 \in D(H_b)$ and

$$H_b f_1 = 0 \quad \text{and} \quad H_b f_2 = 0.$$

When describing properties for h_b, certain natural conditions on the functions $u \in D[h_b]$ will appear. These conditions are

$$\int_b^{b+1} u(x) x^\beta \, dx = 0 \tag{3.1}$$

and

$$\int_b^{b+1} \left(\frac{u(x)}{x^\beta} \right)' x^\alpha \, dx = 0. \tag{3.2}$$

Let us start by looking more closely at the nature of these conditions:

Lemma 3.1. *Let $b > 0$. Then the following hold:*

(i) *If $u \in D[h_b]$ satisfies $u \neq 0$, (3.1) and (3.2), then the functions f_1, f_2 and u are linearly independent.*

(ii) *For any $v \in D[h_b]$, there are constants $c_1, c_2 \in \mathbb{C}$ and a function $u \in D[h_b]$ such that (3.1) and (3.2) hold and such that*

$$v = c_1 f_1 + c_2 f_2 + u.$$

(iii) *Suppose that $F \subset D[h_b]$ in a linear set and has $\dim F \geq 3$. Then there is $u \in F$ with $u \neq 0$ that satisfies (3.1) and (3.2).*

Proof. It will be useful to consider the transformation T defined by

$$(Tv)(x) = \left(\frac{v(x)}{x^\beta} \right)', \quad v \in H^1(b, b+1),$$

as well as the function $g(x) = x^\alpha$. Note that $Tf_1 = 0$ and $Tf_2 = (\alpha + 1)g$.

(i) Let $u \in D[h_b]$ satisfy $u \neq 0$, (3.1) and (3.2). Because of (3.1), it must be that $Tu \neq 0$. Also, since $\int Tu \cdot g\, dx = 0$ by (3.2), the functions Tu and g are linearly independent.

Choose scalars η_1, η_2, η_3 such that $\eta_1 f_1 + \eta_2 f_2 + \eta_3 u = 0$. Then

$$0 = T(\eta_1 f_1 + \eta_2 f_2 + \eta_3 u) = \eta_2(\alpha+1)g + \eta_3 Tu.$$

But since the functions g and Tu are linearly independent, it must be that $\eta_2 = \eta_3 = 0$. It follows that $\eta_1 f_1 = 0$, and thus also $\eta_1 = 0$.

(ii) Choose $v \in D[h_b]$. Let $w = Tv \in L^2(b, b+1)$. Using orthogonal projections one finds that there is a constant c and a function $\tilde{w}$ such that $w = cg + \tilde{w}$ and $(\tilde{w}, g) = 0$. Let

$$\tilde{u}(x) = x^\beta \int_b^x \tilde{w}(t)\, dt \quad \text{and} \quad c_2 = \frac{c}{\alpha+1}.$$

Again by projecting, we find a constant d and a function u such that $\tilde{u} = df_1 + u$ and $(u, f_1) = 0$, that is, u satisfies (3.1). Note that $Tu = T\tilde{u} - dTf_1 = T\tilde{u} = \tilde{w}$. Since $(\tilde{w}, g) = 0$, we get that u satisfies (3.2). Now, $T(c_2 f_2 + u) = cg + \tilde{w} = w$. Hence $T(v - (c_2 f_2 + u)) = 0$, and therefore there must be a constant c_1 such that

$$\frac{v(x) - (c_2 f_2(x) + u(x))}{x^\beta} = c_1.$$

Thus,

$$v = c_1 f_1 + c_2 f_2 + u,$$

and since $f_1, f_2, v \in D[h_b]$, it follows that $u \in D[h_b]$.

(iii) Clearly, F has a two-dimensional subspace G orthogonal to span$\{f_1\}$. This means that every $u \in G$ satisfies (3.1).

Let us prove that TG is two-dimensional. Let $v_1, v_2 \in G$ be linearly independent. Choose $c_1, c_2 \in \mathbb{C}$ such that $c_1 Tv_1 + c_2 Tv_2 = 0$. It follows ny the definition of T that $c_1 v_1 + c_2 v_2 = cf_1$ for some constant $c \in \mathbb{C}$. Since the functions v_1, v_2 and f_1 are linearly independent, it must be that $c_1 = c_2 = 0$. This shows that Tv_1 and Tv_2 are linearly independent. Hence TG is two-dimensional.

In particular there is a $w \in TG$, with $w \neq 0$, orthogonal to g. Find $u \in G$ such that $Tu = w$. Then u satisfies $u \neq 0$ and (3.2). And since $u \in G$, it also satisfies (3.1). ☐

Lemma 3.2. *Let $0 \le \nu < 3$ and $\nu \le 2\beta$. There is a constant $C_\nu > 0$ such that*

$$\sup_{b \le y \le b+1} \frac{|u(y)|^2}{y^\nu} \le C_\nu h_b[u],$$

for any $b > 0$ and any $u \in D[h_b]$ that satisfies (3.1) *and* (3.2).

Proof. The proof is divided into two cases. The first case is when $0 < b \le 1$ and the second case is when $b > 1$.

Case 1. Choose b with $0 < b \leq 1$ and $u \in D\left[h_b\right]$ with $u \neq 0$, that satisfies (3.1) and (3.2). Also choose y with $b \leq y \leq b+1$. Clearly,

$$\begin{aligned}\frac{u(y)}{y^{\beta}}\left(y^{2\beta+1}-b^{2\beta+1}\right) &= \int_b^y\left(\frac{u(t)}{t^{\beta}}\left(t^{2\beta+1}-b^{2\beta+1}\right)\right)' dt \\ &= (2\beta+1)\int_b^y u(t)t^{\beta}\, dt + \int_b^y\left(\frac{u(t)}{t^{\beta}}\right)'\left(t^{2\beta+1}-b^{2\beta+1}\right) dt,\end{aligned}$$

and also

$$\begin{aligned}\frac{u(y)}{y^{\beta}}\left((b+1)^{2\beta+1}-y^{2\beta+1}\right) &= -\int_y^{b+1}\left(\frac{u(t)}{t^{\beta}}\left((b+1)^{2\beta+1}-t^{2\beta+1}\right)\right)' dt \\ &= (2\beta+1)\int_y^{b+1} u(t)t^{\beta}\, dt - \int_y^{b+1}\left(\frac{u(t)}{t^{\beta}}\right)'\left((b+1)^{2\beta+1}-t^{2\beta+1}\right) dt.\end{aligned}$$

Hence, using (3.1),

$$\frac{u(y)}{y^{\beta}} = \Gamma_1\left(\int_b^y\left(\frac{u(t)}{t^{\beta}}\right)'\left(t^{2\beta+1}-b^{2\beta+1}\right) dt \right. \tag{3.3}$$

$$\left. - \int_y^{b+1}\left(\frac{u(t)}{t^{\beta}}\right)'\left((b+1)^{2\beta+1}-t^{2\beta+1}\right) dt\right) \tag{3.4}$$

where

$$\Gamma_1 = \frac{1}{(b+1)^{2\beta+1}-b^{2\beta+1}}.$$

Similarly to the above, using (3.2), we find that for any t with $b \leq t \leq b+1$,

$$\frac{1}{t^{\alpha}}\left(\frac{u(t)}{t^{\beta}}\right)' = \Gamma_2\left(\int_b^t\left(\frac{1}{x^{\alpha}}\left(\frac{u(x)}{x^{\beta}}\right)'\right)'\left(x^{2\alpha+1}-b^{2\alpha+1}\right) dx \right. \tag{3.5}$$

$$\left. - \int_t^{b+1}\left(\frac{1}{x^{\alpha}}\left(\frac{u(x)}{x^{\beta}}\right)'\right)'\left((b+1)^{2\beta+1}-x^{2\beta+1}\right) dx\right), \tag{3.6}$$

where

$$\Gamma_2 = \frac{1}{(b+1)^{2\alpha+1}-b^{2\alpha+1}}.$$

Combine (3.3) and (3.5) to obtain

$$\frac{u(y)}{y^{\nu/2}} = \Gamma_1\Gamma_2 \cdot y^{\beta-\frac{\nu}{2}}\Big(I_1 - I_2 - I_3 + I_4\Big),$$

where

$$I_1 = \int_b^y\!\!\int_b^t\left(\frac{1}{x^{\alpha}}\left(\frac{u(x)}{x^{\beta}}\right)'\right)'\left(x^{2\alpha+1}-b^{2\alpha+1}\right) t^{\alpha}\left(t^{2\beta+1}-b^{2\beta+1}\right) dx\, dt,$$

$$I_2 = \int_y^{b+1}\!\!\int_b^t\left(\frac{1}{x^{\alpha}}\left(\frac{u(x)}{x^{\beta}}\right)'\right)'\left(x^{2\alpha+1}-b^{2\alpha+1}\right) t^{\alpha}\left((b+1)^{2\beta+1}-t^{2\beta+1}\right) dx\, dt,$$

$$I_3 = \int_b^y\!\!\int_t^{b+1}\!\left(\frac{1}{x^\alpha}\left(\frac{u(x)}{x^\beta}\right)'\right)'\!\left((b+1)^{2\alpha+1}-x^{2\alpha+1}\right)t^\alpha\left(t^{2\beta+1}-b^{2\beta+1}\right)\,dx\,dt$$

and

$$I_4 = \int_y^{b+1}\!\!\int_t^{b+1}\!\left(\frac{1}{x^\alpha}\left(\frac{u(x)}{x^\beta}\right)'\right)'\!\left((b+1)^{2\alpha+1}-x^{2\alpha+1}\right)t^\alpha\left((b+1)^{2\beta+1}-t^{2\beta+1}\right)\,dx\,dt.$$

It remains to find constants $C_1, C_2, C_3, C_4 > 0$, independent of b, u and y, such that

$$\left|y^{\beta-\frac{\nu}{2}} I_j\right|^2 \le C_j h_b[u], \quad j = 1,2,3,4,$$

since then we can use the fact that $\Gamma_1 \le 1$ and $\Gamma_2 \le 1$, to conclude that

$$\frac{|u(y)|^2}{y^\nu} \le 4\,(C_1 + C_2 + C_3 + C_4)\,h_b[u].$$

Note that by the Cauchy-Schwarz inequality,

$$\left|y^{\beta-\frac{\nu}{2}} I_1\right|^2 \le J_1(b,y)^2 \cdot h_b[u]$$

where

$$J(b,y) = y^{\beta-\frac{\nu}{2}} \int_b^y t^\alpha\left(t^{2\beta+1}-b^{2\beta+1}\right)\left(\int_b^t \frac{\left(x^{2\alpha+1}-b^{2\alpha+1}\right)^2}{x^{2(\alpha+\beta)}}\,dx\right)^{1/2} dt,$$

and similarly for I_2, I_3 and I_4. Now, as soon as $\alpha \ge 0$, $0 \le \beta < 3/2 + \alpha$, $0 \le \nu < 3$ and $\nu \le 2\beta$, some calculations show that there exists constants $C_1, C_2, C_3, C_4 > 0$ such that for any $b > 0$ and y with $b \le y \le b+1$

$$J_n(b,y)^2 \le C_n, \quad n = 1,2,3,4.$$

Case 2. Choose $b > 1$, $u \in D\,[h_b]$ that satisfies (3.1) and (3.2), and $y \in [b, b+1]$. Set $v(x) = u(x)/x^\beta$. By (3.1) and (3.2), there are points $x_1, x_2 \in [b, b+1]$ such that

$$v(x_1) = 0 \quad \text{and} \quad v'(x_2) = 0.$$

Let us start by showing that

$$\left|\frac{v'(t)}{t^\alpha}\right|^2 \le h_b[u]\frac{1}{b^{2(\alpha+\beta)}}, \quad b \le t \le b+1. \tag{3.7}$$

Choose t with $b \le t \le b+1$. Assume first that $b \ge x_2$. Note that

$$\frac{v'(t)}{t^\alpha} = \int_{x_2}^t \left(\frac{v'(x)}{x^\alpha}\right)' x^{\alpha+\beta} \cdot \frac{1}{x^{\alpha+\beta}}\,dx.$$

Hence,

$$\left|\frac{v'(t)}{t^\alpha}\right|^2 \le \int_{x_2}^t \left|\left(\frac{v'(t)}{x^\alpha}\right)' x^{\alpha+\beta}\right|^2 dx \cdot \int_{x_2}^t \frac{1}{x^{2(\alpha+\beta)}}\,dx \le h_b[u] \cdot \frac{1}{b^{2(\alpha+\beta)}}$$

The case when $b \le x_2$ is handled in a similar way. This concludes the proof of equation (3.7).

Now, assume that $y \ge x_1$. It follows from (3.7) that

$$\begin{aligned}|v(y)|^2 &= \left|\int_{x_1}^{y} \frac{v'(t)}{t^\alpha} \cdot t^\alpha \, dt\right|^2 \\ &\le \int_{x_1}^{y} \left|\frac{v'(t)}{t^\alpha}\right|^2 dt \cdot \int_{x_1}^{y} t^{2\alpha} \, dt \\ &\le h_b[u] \cdot \frac{1}{b^{2(\alpha+\beta)}} \cdot (b+1)^{2\alpha}.\end{aligned}$$

Therefore, since $b > 1$

$$\frac{u(y)}{y^\nu} \le \frac{1}{y^\nu} \cdot \frac{(b+1)^{2(\alpha+\beta)}}{b^{2(\alpha+\beta)}} \cdot h_b[u] \le 2^{2(\alpha+\beta)} \cdot h_b[u].$$

By similar arguments, this inequality also holds when $y \le x_1$. □

Proposition 3.3. *Let $0 \le \nu < 3$ and $\nu \le 2\beta$. There are constants $D_\nu, E_\nu > 0$ such that for any $b > 0$ and any non-negative bounded potential $V \ne 0$ that satisfies*

$$\int_b^{b+1} V(x) x^\nu \, dx \le D_\nu, \tag{3.8}$$

the operator $H_b - V$ has negative spectrum consisting of exactly two eigenvalues, $-E_1$ and $-E_2$, that satisfy

$$E_1 \le E_\nu \quad \textit{and} \quad E_2 \le E_\nu.$$

Proof. For given $b > 0$, let

$$g_b = f_2 - \frac{\int_b^{b+1} f_1 f_2 \, dx}{\int_b^{b+1} |f_1|^2 \, dx} \cdot f_1,$$

so that f_1 and g_b are orthogonal in $L^2(b, b+1)$ and span the same subspace as f_1 and f_2. Introduce

$$B_1 = \frac{1}{4} \inf_{b>0} \left(\frac{\int_b^{b+1} |f_1(x)|^2 \, dx}{\sup\limits_{b \le y \le b+1} \frac{|f_1(y)|^2}{y^\nu}} \right) \quad \text{and} \quad B_2 = \frac{1}{4} \inf_{b>0} \left(\frac{\int_b^{b+1} |g_b(x)|^2 \, dx}{\sup\limits_{b \le y \le b+1} \frac{|g_b(y)|^2}{y^\nu}} \right).$$

Elementary calculations show that $B_1 > 0$ and $B_2 > 0$. Now set

$$B = \min\{B_1, B_2\}.$$

Let $C_\nu > 0$ be as in Lemma 3.2, and

$$D_\nu = \min\left\{ \frac{B}{2C_\nu(1+B)}, \frac{B}{2C_0(1+B)}, \frac{1}{C_\nu} \right\} \quad \text{and} \quad E_\nu = \frac{1}{C_0(1+B)}.$$

Note that this in particular implies

$$C_\nu D_\nu \le 1, \tag{3.9}$$

$$C_0 E_\nu + 2C_\nu D_\nu \leq 1 \tag{3.10}$$

and

$$BE_\nu \geq 2D_\nu. \tag{3.11}$$

Choose $b > 0$ and a non-negative $V \neq 0$ that satisfies condition (3.8).

Step 1. Let us start by showing that $H_b - V \geq -E_\nu$, in the sense of quadratic forms. Choose $v \in D[h_b]$. By (ii) of Lemma 3.1 there are constants $c_1, c_2 \in \mathbb{C}$ and a function $u \in D[h_b]$ for which (3.1) and (3.2) hold, such that

$$v = c_1 f_1 + c_2 g_b + u.$$

Write $w = c_1 f_1 + c_2 g_b$ and observe that

$$\begin{aligned}
\frac{E_\nu}{2} \int_b^{b+1} |w(x)|^2 \, dx &= \frac{E_\nu}{2} \left(|c_1|^2 \int_b^{b+1} |f_1(x)|^2 \, dx + |c_2|^2 \int_b^{b+1} |g_b(x)|^2 \, dx \right) \\
&\geq E_\nu \left(B_1 \cdot 2|c_1|^2 \sup_{b \leq y \leq b+1} \frac{|f_1(y)|^2}{y^\nu} + B_2 \cdot 2|c_2|^2 \sup_{b \leq y \leq b+1} \frac{|g_b(y)|^2}{y^\nu} \right) \\
&\geq BE_\nu \cdot \sup_{b \leq y \leq b+1} \frac{|w(y)|^2}{y^\nu}.
\end{aligned}$$

In other words, using (3.11),

$$\frac{E_\nu}{2} \int_b^{b+1} |w(x)|^2 \, dx - 2D_\nu \sup_{b \leq y \leq b+1} \frac{|w(y)|^2}{y^\nu} \geq 0. \tag{3.12}$$

Consider the quadratic expression

$$g[v] = h_b[v] - \int_b^{b+1} V(x)|v(x)|^2 \, dx + E_\nu \int_b^{b+1} |v(x)|^2 \, dx.$$

We have to prove that $g[v] \geq 0$. Note that $h_b[v] = h_b[u]$. Then use the facts that $|a+b|^2 \leq 2|a|^2 + 2|b|^2$ and $|a+b|^2 \geq \frac{1}{2}|a|^2 - |b|^2$, for any $a, b \in \mathbb{C}$, together with (3.8), (3.12) and Lemma 3.2, to get that

$$\begin{aligned}
g[v] &\geq h_b[u] - \int_b^{b+1} (E_\nu + 2V(x)) \, |u(x)|^2 \, dx + \int_b^{b+1} \left(\frac{E_\nu}{2} - 2V(x) \right) |w(x)|^2 \, dx \\
&\geq h_b[u] - \sup_{b \leq y \leq b+1} |u(y)|^2 \cdot E_\nu - 2 \sup_{b \leq y \leq b+1} \frac{|u(y)|^2}{y^\nu} \cdot D_\nu \\
&\quad + \frac{E_\nu}{2} \int_b^{b+1} |w(x)|^2 \, dx - 2D_\nu \sup_{b \leq y \leq b+1} \frac{|w(y)|^2}{y^\nu} \\
&\geq h_b[u] \left(1 - (C_0 E_\nu + 2C_\nu D_\nu)\right) \geq 0.
\end{aligned}$$

Step 2. Continue by showing that the negative spectrum of $H_b - V$ is discrete and consists of at most two eigenvalues. Let E be the spectral measure corresponding to $H_b - V$, and denote by $N_-(H_b - V)$ the rank of $E(-\infty, 0)$. From Glazman's lemma, see Remark 3.4, it is known that

$$N_-(H_b - V) = \sup_F \dim F, \tag{3.13}$$

where the supremum is taken over all linear subsets $F \subset D\,[h_b]$ such that

$$h_b[f] - \int_b^{b+1} V|f|^2\,dx < 0,$$

for any $f \in F$ with $f \neq 0$. Suppose that $F \subset D\,[h_b]$ is a linear set that satisfies $\dim F \geq 3$. By (iii) of Lemma 3.1 we find a $u \in F$ with $u \neq 0$ that satisfies (3.1) and (3.2). It follows from Lemma 3.2, (3.8) and (3.9) that

$$\begin{aligned} h_b[u] - \int_b^{b+1} V(x)|u(x)|^2\,dx &\geq h_b[u] - \sup_{b \leq y \leq b+1} \frac{|u(y)|^2}{y^\nu} \int_b^{b+1} V(x) x^\nu\,dx \\ &\geq h_b[u] - C_\nu D_\nu h_b[u] \\ &\geq 0. \end{aligned}$$

Using (3.13), this shows that $N_-(H_b - V) \leq 2$.

Step 3. The final step is to provide the existence of at least two negative eigenvalues of $H_b - V$. Let $F = \operatorname{span}\{f_1, f_2\}$. Clearly $h_b[f] = 0$ for any $f \in F$. In particular

$$h_b[f] - \int_b^{b+1} V|f|^2\,dx < 0,$$

for $f \in F$ with $f \neq 0$. Again using (3.13), it is seen that $N_-(H_b - V) \geq 2$. $\square$

Remark 3.4 (Glazman's lemma)**.** Much of the variational techniques used here origin in *Glazman's lemma*. This lemma, discussed in e.g. [BS92] states the following:

Let $\mathscr{H}$ be a separable Hilbert space, and D a dense, linear subset of $\mathscr{H}$. Suppose that the semi-bounded quadratic form a is closable on D, and denote by A the self-adjoint operator corresponding to the closure of a. Also let E be the spectral measure corresponding to A. Then

$$\operatorname{rank} E(-\infty, x) = \sup \dim F,$$

where the supremum if taken over all linear subsets $F \subset D$ such that

$$a[f] < x\|f\|^2, \quad f \in F, f \neq 0.$$

4 Well-Behaved Potentials

In this section we restrict ourselves to bounded potentials V of compact support. One obvious reason for this is to avoid difficulties in defining operators

$$H = H_0 - V, \quad \text{where} \quad H_0 = (-\Delta)^2 - C_{d,2}^{\mathrm{HR}} \frac{1}{|x|^4}.$$

Indeed, when V is bounded, this operator is simply defined as an operator sum with domain $D\,(H) = D\,(H_0)$.

4.1 General Half-Line Case

We start by proving the general half-line result that follows from Proposition 3.3. The proof uses techniques from [Wei96] and [EF08].

Proposition 4.1. *Let $a \geq 0$, $\beta \geq 0$ and $0 \leq \nu < 3$, $\nu \leq 2\beta$. Define the quadratic form h_0 as the closure of*

$$u \mapsto \int_0^\infty \left| \frac{d}{dx}\left(\frac{1}{x^\alpha}\frac{d}{dx}\left(\frac{u(x)}{x^\beta}\right)\right)\right|^2 x^{2(\alpha+\beta)}\, dx,$$

on $C_0^\infty(0,\infty)$. Let H_0 be the self-adjoint operator in $L^2(0,\infty)$ corresponding to h_0. Then there is a constant $C = C(\alpha,\beta,\nu) > 0$ such that for any non-negative bounded potential V with compact support in $(0,\infty)$, the negative spectrum of $H_0 - V$ is discrete and

$$\operatorname{tr}(H_0 - V)_-^{\frac{3-\nu}{4}} \leq C \int_0^\infty V(x) x^\nu\, dx.$$

Proof. Let D_ν and E_ν be as in Proposition 3.3. Choose a bounded $V \geq 0$ with $\operatorname{supp} V \Subset (0,\infty)$. Define a sequence of numbers $a_1 < a_2 < \cdots$ by setting $a_1 = \min \operatorname{supp} V > 0$ and recursively,

$$(a_{j+1} - a_j)^{3-\nu} \int_{a_j}^{a_{j+1}} V(x) x^\nu\, dx = D_\nu, \tag{4.1}$$

for $j \geq 2$. The recursion stops for $j = n$, when $a_n \geq \max \operatorname{supp} V$. Indeed, the sequence is finite, since

$$a_{j+1} - a_j \geq \left(\frac{D_\nu}{\int_0^\infty V(x) x^\nu\, dx}\right)^{\frac{1}{3-\nu}}$$

for any j. Let $a_0 = 0$ and $a_{n+1} = \infty$.

For $0 \leq j \leq n$, consider the quadratic forms

$$g_j[v] = \int_{a_j}^{a_{j+1}} \left| \frac{d}{dx}\left(\frac{1}{x^\alpha}\frac{d}{dx}\left(\frac{v(x)}{x^\beta}\right)\right)\right|^2 x^{2(\alpha+\beta)}\, dx.$$

For $1 \leq j \leq n$, the domain of g_j is $D[g_j] = H^2(a_j, a_{j+1})$. For $j = 0$, it can be shown that g_0 is closable on $C_0^\infty((0, a_1])$, and we consider it as the closure on this domain. In this way all the forms g_j are closed. Let G_j be the self-adjoint operator in $L^2(a_j, a_{j+1})$ corresponding to g_j.

By comparing quadratic forms and their domains of definition, we get that

$$H_0 - V \geq \bigoplus_{j=0}^{n} (G_j - V).$$

Note that $G_j - V \geq 0$ for $j = 0$ and for $j = n$, since $V = 0$ on (a_j, a_{j+1}) for $j = 0, n$. Therefore, if we can prove that the negative spectrum of $G_j - V$ is

discrete for $j = 1, 2, \ldots, n-1$, it follows that the negative spectrum of $H_0 - V$ is also discrete. In this case,

$$\operatorname{tr}(H_0 - V)_-^{\frac{3-\nu}{4}} \leq \sum_{j=1}^{n-1} \operatorname{tr}(G_j - V)_-^{\frac{3-\nu}{4}}. \tag{4.2}$$

Let $1 \leq j \leq n-1$ and define a unitary transformation $\mathscr{U}$ from $L^2(a_j, a_{j+1})$ onto $L^2(b_j, b_j + 1)$ by

$$(\mathscr{U} v)(x) = (a_{j+1} - a_j)^{1/2} v\left((a_{j+1} - a_j)x\right),$$

where $b_j = a_j/(a_{j+1} - a_j)$. Clearly,

$$G_j - V = \frac{1}{(a_{j+1} - a_j)^4} \mathscr{U}^{-1} \left(H_{b_j} - V_j\right) \mathscr{U},$$

where H_b is the operator discussed in Section 3 and where

$$V_j(x) = (a_{j+1} - a_j)^4 V\left((a_{j+1} - a_j)x\right).$$

Note that by (4.1),

$$\int_{b_j}^{b_j+1} V_j(x) x^\nu \, dx = D_\nu.$$

Hence by Proposition 3.3, the negative spectrum of $H_{b_j} - V_j$, an therefore also of $G_j - V$, is discrete. By the same proposition and (4.1),

$$\begin{aligned} \operatorname{tr}(G_j - V)_-^{\frac{3-\nu}{4}} &= \frac{1}{(a_{j+1} - a_j)^{3-\nu}} \operatorname{tr}\left(H_{b_j} - V_j\right)_-^{\frac{3-\nu}{4}} \\ &\leq \frac{2}{(a_{j+1} - a_j)^{3-\nu}} E_\nu^{\frac{3-\nu}{4}} \\ &= \frac{2E_\nu^{\frac{3-\nu}{4}}}{D_\nu} \int_{a_j}^{a_{j+1}} V(x) x^\nu \, dx. \end{aligned}$$

We have thus shown that the negative spectrum of $H_0 - V$ is discrete, and by (4.2), that

$$\operatorname{tr}(H_0 - V)_-^{\frac{3-\nu}{4}} \leq \frac{2E_\nu^{\frac{3-\nu}{4}}}{D_\nu} \int_0^\infty V(x) x^\nu \, dx. \qquad \square$$

By standard methods, as described in [Hun07], and originally in [AL78], this result extends to an inequality for $\operatorname{tr}(H_0 - V)_-^\gamma$ for any $\gamma \geq \gamma_{\mathrm{c}}$, where

$$\gamma_{\mathrm{c}} = \frac{3-\nu}{4}.$$

Corollary 4.2. *Let α, β, ν and H_0 be as in Proposition 4.1. Then, for any $\gamma \geq \gamma_{\mathrm{c}}$, there is a $C = C(\alpha, \beta, \nu, \gamma) > 0$ such that for any non-negative, bounded potential V with compact support in $(0, \infty)$,*

$$\operatorname{tr}(H_0 - V)_-^\gamma \leq C \int_0^\infty V(x)^{1+\gamma-\gamma_{\mathrm{c}}} x^\nu \, dx.$$

4.2 The Fourth-Order Operator on the Half-Line

We now consider the fourth-order operator $d^4/dx^4 - C_{1,2}^{\mathrm{HR}}/x^4 - V(x)$ in $L^2(0,\infty)$, where $C_{1,2}^{\mathrm{HR}} = 9/16$ is the sharp constant in the classical Hardy-Rellich inequality

$$\int_0^\infty |u''(x)|^2\,dx \geq C_{1,2}^{\mathrm{HR}} \int_0^\infty \frac{|u(x)|^2}{x^4}\,dx, \quad u \in C_0^\infty(0,\infty)\,.$$

Proposition 4.3. *Let $0 \leq \nu < 3$ and consider the operator*

$$H_0 = \frac{d^4}{dx^4} - C_{1,2}^{\mathrm{HR}} \frac{1}{x^4}$$

in $L^2(0,\infty)$ defined as the Friedrich extension of the corresponding operator initially defined on $C_0^\infty(0,\infty)$. Then, for any $\gamma \geq (3-\nu)/4$, there is a $C = C(\nu,\gamma) > 0$ such that for any non-negative, bounded potential V with compact support in $(0,\infty)$, the negative spectrum of $H_0 - V$ is discrete and

$$\operatorname{tr}(H_0 - V)_-^\gamma \leq C \int_0^\infty V(x)^{\gamma + \frac{1+\nu}{4}} x^\nu\,dx.$$

Proof. Note that since $C_{1,2}^{\mathrm{HR}} = 9/16$, the closed quadratic form h_0 corresponding to H_0 is the closure of

$$u \mapsto \int_0^\infty \left(|u''(x)|^2 - \frac{9}{16} \cdot \frac{|u(x)|^2}{x^4} \right) dx$$

on $C_0^\infty(0,\infty)$. By partial integration we see that for $u \in C_0^\infty(0,\infty)$,

$$h_0[u] = \int_0^\infty \left| \frac{d}{dx} \left(\frac{1}{x^\alpha} \frac{d}{dx} \left(\frac{u(x)}{x^\beta} \right) \right) \right|^2 x^{2(\alpha+\beta)}\,dx,$$

where

$$\alpha = \frac{\sqrt{10}-2}{2} \quad \text{and} \quad \beta = \frac{3}{2}.$$

Hence the result follows from Corollary 4.2. $\square$

4.3 The Fourth-Order Operator in Three Dimensions

Let us turn to the operator $(-\Delta)^2 - C_{3,2}^{\mathrm{HR}}/|x|^4 - V(x)$ in $L^2(\mathbb{R}^3)$. In $\mathbb{R}^3$ we have the Hardy-Rellich inequality

$$\int_{\mathbb{R}^3} |\Delta u(x)|^2\,dx \geq C_{3,2}^{\mathrm{HR}} \int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x|^4}\,dx, \quad u \in C_0^\infty\left(\mathbb{R}^3 \backslash \{0\}\right),$$

where the sharp constant $C_{3,2}^{\mathrm{HR}}$ conveniently enough coincides with $C_{1,2}^{\mathrm{HR}} = 9/16$. We denote by S^2 and σ the unit sphere and two-dimensional surface measure in $\mathbb{R}^3$, respectively. Let Y_n, $n = 0,1,2,\ldots$ be the normalized eigenfunctions of the Laplace-Beltrami operator in $L^2(S^2,\sigma)$. The eigenfunction Y_n corresponds to

the eigenvalue $n(n+1)$, and in particular Y_0 is constant. Consider the canonical isometric isomorphism

$$\mathscr{U} : L^2\left(\mathbb{R}^3\right) \to \bigoplus_{n=0}^{\infty} L^2\left(0,\infty\right)$$

given by

$$\mathscr{U} u = \{u_n\}_{n=0}^{\infty}, \qquad \text{where} \qquad u_n(r) = r\int_{S^2} u(r\theta)Y_n(\theta)\,d\sigma(\theta)$$

for $u \in C_0^\infty\left(\mathbb{R}^3\backslash\{0\}\right)$. For such u, it is the case that $u_n \in C_0^\infty(0,\infty)$ and

$$\mathscr{U}(-\Delta u) = \left\{-u_n''(r) + n(n+1)\frac{u_n(r)}{r^2}\right\}_{n=0}^{\infty}. \tag{4.3}$$

Finally, let $\mathscr{H} = \bigoplus_{n=0}^{\infty} L^2\left(0,\infty\right)$ and consider the orthogonal decomposition $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$, where

$$\mathscr{H}_1 = \{\{u_n\}_{n=0}^{\infty}\,;\, u_n = 0 \text{ for } n \geq 1\} \qquad \text{and} \qquad \mathscr{H}_2 = \{\{u_n\}_{n=0}^{\infty}\,;\, u_0 = 0\}.$$

Let P_1 and P_2 be the orthogonal projections in $\mathscr{H}$ onto the subspaces $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. Depending on context, we will sometimes identify $\mathscr{H}_1$ with the space $L^2\left(0,\infty\right)$.

Lemma 4.4. *For any $u \in C_0^\infty(0,\infty)$ and any $c > 0$,*

$$\int_0^\infty \left|-u''(x) + c\frac{u(x)}{x^2}\right|^2 dx \geq \left(c^2 - \frac{3}{2}c + C_{1,2}^{\mathrm{HR}}\right)\int_0^\infty \frac{|u(x)|^2}{x^4}\,dx.$$

Proof. Choose $u \in C_0^\infty(0,\infty)$ and any $c > 0$. Recall the classical Hardy-Rellich inequalities

$$\int_0^\infty |u'(x)|^2\,dx \geq \frac{1}{4}\int_0^\infty \frac{|u(x)|^2}{x^2}\,dx \tag{4.4}$$

and

$$\int_0^\infty |u''(x)|^2\,dx \geq C_{1,2}^{\mathrm{HR}}\int_0^\infty \frac{|u(x)|^2}{x^4}\,dx. \tag{4.5}$$

It follows from (4.4) that

$$\int_0^\infty \frac{|u'(x)|^2}{x^2}\,dx \geq \frac{9}{4}\int_0^\infty \frac{|u(x)|^2}{x^4}\,dx. \tag{4.6}$$

Now, by partial integration,

$$\int_0^\infty \left|-u''(x) + c\frac{u(x)}{x^2}\right|^2 dx = \int_0^\infty\left(|u''(x)|^2 + 2c\frac{|u'(x)|^2}{x^2} + (c^2{-}6c)\frac{|u(x)|^2}{x^4}\right) dx$$

Combine this with (4.5) and (4.6) to obtain the result. ◻

Lemma 4.5. *Let $\gamma \geq 1/4$ and let $G_0^{(1)}$ be the self-adjoint operator in $\mathscr{H}_1 \cong L^2(0,\infty)$ that corresponds to the closure $g_0^{(1)}$ of the quadratic form*

$$u \mapsto \int_0^\infty \left(|u''(r)|^2 - C_{1,2}^{\mathrm{HR}} \frac{|u(r)|^2}{r^4} \right) dr,$$

initially defined on $C_0^\infty(0,\infty)$. Then there is a constant $C^{(1)} = C^{(1)}(\gamma) > 0$ such that given a non-negative $V \in C_0^\infty\left(\mathbb{R}^3 \backslash \{0\}\right)$ and $V^{(1)} = P_1 \mathscr{U} V \mathscr{U}^{-1} P_1$, the negative spectrum of $G_0 - V^{(1)}$ is discrete and

$$\operatorname{tr}\left(G_0^{(1)} - V^{(1)}\right)_-^\gamma \leq C^{(1)} \int_{\mathbb{R}^3} V(x)^{\gamma + \frac{3}{4}}\, dx.$$

Proof. Use Proposition 4.3 with $\nu = 2$ to obtain a constant $C = C(\gamma) > 0$ such that for any non-negative $W \in C_0^\infty(0,\infty)$, the negative spectrum of $G_0^{(1)} - W$ is discrete and

$$\operatorname{tr}\left(G_0^{(1)} - W\right)_-^\gamma \leq C \int_0^\infty W(r)^{\gamma + \frac{3}{4}}\, r^2\, dr.$$

Note that for any non-negative $V \in C_0^\infty\left(\mathbb{R}^3 \backslash \{0\}\right)$, the operator $V^{(1)}$ is simply multiplication with the function $\tilde{V} \in C_0^\infty(0,\infty)$, where

$$\tilde{V}(r) = \frac{1}{\sigma(S^2)} \int_{S^2} V(r\theta)\, d\sigma(\theta), \quad r > 0.$$

In particular, the negative spectrum of $G_0^{(1)} - V^{(1)}$ is discrete and

$$\begin{aligned} \operatorname{tr}\left(G_0^{(1)} - V^{(1)}\right)_-^\gamma C \int_0^\infty \tilde{V}(r)^{\gamma + \frac{3}{4}}\, r^2\, dr \\ \leq \frac{C}{\sigma(S^2)} \int_{\mathbb{R}^3} V(x)^{\gamma + \frac{3}{4}}\, dx, \end{aligned}$$

since by Hölder's inequality,

$$\tilde{V}(r)^{\gamma + \frac{3}{4}} \leq \frac{1}{\sigma(S^2)} \int_{S^2} V(r\theta)^{\gamma + \frac{3}{4}}\, d\sigma(\theta), \quad r > 0. \qquad \square$$

We will use the following fourth-order Lieb-Thirring inequality in $L^2\left(\mathbb{R}^3\right)$, that follows from more general results in [NW96]. The operator $(-\Delta)^2$ is of course defined as the self-adjoint operator in $L^2\left(\mathbb{R}^3\right)$ corresponding to the closure of the quadratic form

$$u \mapsto \int_{\mathbb{R}^3} |\Delta u(x)|^2\, dx, \quad u \in C_0^\infty\left(\mathbb{R}^3 \backslash \{0\}\right).$$

Lemma 4.6. *For any $\gamma \geq 1/4$, there is a constant $D = D(\gamma) > 0$ such that for any non-negative $V \in C_0^\infty\left(\mathbb{R}^3 \backslash \{0\}\right)$, the negative spectrum of $(-\Delta)^2 - V$ is discrete and*

$$\operatorname{tr}\left((-\Delta)^2 - V\right)_-^\gamma \leq D \int_{\mathbb{R}^3} V(x)^{\gamma + \frac{3}{4}}\, dx.$$

Lemma 4.7. *Let $\gamma \geq 1/4$ and denote by $G_0^{(2)}$ the self-adjoint operator in $\mathscr{H}_2$ corresponding to the closure $g_0^{(2)}$ of the quadratic form*

$$\{u_n\}_{n=1}^{\infty} \mapsto \sum_{n=1}^{\infty} \int_0^{\infty} \left| -u_n''(r) + n(n+1)\frac{u_n(r)}{r^2} \right|^2 dr,$$

initially defined on $D := P_2 \mathscr{U} C_0^{\infty}\left(\mathbb{R}^3 \backslash \{0\}\right)$. Then there is a constant $C^{(2)} > 0$ such given a non-negative $V \in C_0^{\infty}\left(\mathbb{R}^3 \backslash \{0\}\right)$ and $V^{(2)} = P_2 \mathscr{U} V \mathscr{U}^{-1} P_2$, the negative spectrum of $G_0^{(2)} - V^{(2)}$ is discrete and

$$\operatorname{tr}\left(G_0^{(2)} - V^{(2)}\right)_-^{\gamma} \leq C^{(2)} \int_{\mathbb{R}^3} V(x)^{\gamma + \frac{3}{4}}\, dx.$$

Proof. Consider the operator $(-\Delta)^2$ in $L^2\left(\mathbb{R}^3\right)$. Let the constant $D > 0$ be as in Lemma 4.6. By (4.3), the operator $\hat{G}_0^{(2)} := \mathscr{U}(-\Delta)^2 \mathscr{U}^{-1}$ corresponds to the closure in $\mathscr{H}$ of the quadratic form

$$\{u_n\}_{n=0}^{\infty} \mapsto \sum_{n=0}^{\infty} \int_0^{\infty} \left| -u_n''(r) + n(n+1)\frac{u_n(r)}{r^2} \right|^2 dr,$$

initially defined on $\hat{D} := \mathscr{U} C_0^{\infty}\left(\mathbb{R}^3 \backslash \{0\}\right)$.

Choose $V \in C_0^{\infty}\left(\mathbb{R}^3 \backslash \{0\}\right)$ and let $V^{(2)} = P_2 \mathscr{U} V \mathscr{U}^{-1} P_2$ and $\hat{V}^{(2)} = \mathscr{U} V \mathscr{U}^{-1}$. Denote by E and $\hat{E}$ the spectral measures corresponding to the operators $G_0^{(2)} - V^{(2)}$ and $\hat{G}_0^{(2)} - \hat{V}^{(2)}$, respectively. By Glazman's lemma,

$$\operatorname{rank} E(-\infty, -\lambda) \leq \operatorname{rank} \hat{E}(-\infty, -\lambda), \tag{4.7}$$

for any $\lambda > 0$. Lemma 4.6 shows that the negative spectrum of $\hat{G}_0^{(2)} - \hat{V}^{(2)}$ is discrete and that

$$\operatorname{tr}\left(\hat{G}_0^{(2)} - \hat{V}^{(2)}\right)_-^{\gamma} \leq D \int_{\mathbb{R}^3} V(x)^{\gamma + \frac{3}{4}}\, dx.$$

The result follows from this and (4.7). ∎

Proposition 4.8. *Define the quadratic form h_0 as the closure of*

$$u \mapsto \int_{\mathbb{R}^3} \left(|\Delta u(x)|^2 - C_{3,2}^{\mathrm{HR}} \frac{|u(x)|^2}{|x|^4} \right) dx$$

on $C_0^{\infty}\left(\mathbb{R}^3 \backslash \{0\}\right)$. Let H_0 be the self-adjoint operator in $L^2\left(\mathbb{R}^3\right)$ corresponding to h_0. Then there is a constant $C > 0$ such that for any non-negative $V \in C_0^{\infty}\left(\mathbb{R}^3 \backslash \{0\}\right)$, the negative spectrum of $H_0 - V$ is discrete and

$$\operatorname{tr}(H_0 - V)_-^{1/4} \leq C \int_{\mathbb{R}^3} V(x)\, dx.$$

Proof. Let $G_0^{(1)}$, $G_0^{(2)}$, $g_0^{(1)}$, $g_0^{(2)}$, $C^{(1)}$ and $C^{(2)}$ be as in Lemmas 4.5 and 4.7. Choose any $u \in C_0^{\infty}\left(\mathbb{R}^3 \backslash \{0\}\right)$ and ϵ with $0 < \epsilon < 1$. Write $\{u_n\}_{n=0}^{\infty} = \mathscr{U} u$

and note that each $u_n \in C_0^\infty(0,\infty)$. Using (4.3) and $C_{3,2}^{\mathrm{HR}} = C_{1,2}^{\mathrm{HR}}$, and finally Lemma 4.4 it follows that

$$\begin{aligned}
h_0[u] &= \sum_{n=0}^{\infty} \int_0^\infty \left(\left| -u_n''(r) + n(n+1)\frac{u_n(r)}{r^2} \right|^2 - C_{1,2}^{\mathrm{HR}} \frac{|u_n(r)|^2}{r^4} \right) dr \\
&\geq \int_0^\infty \left(|u_0''(r)|^2 - C_{1,2}^{\mathrm{HR}} \frac{|u_0(r)|^2}{r^4} \right) dr \\
&\quad + \epsilon \sum_{n=1}^{\infty} \int_0^\infty \left| -u_n''(r) + n(n+1)\frac{u_n(r)}{r^2} \right|^2 dr \\
&\quad + \sum_{n=1}^{\infty} \int_0^\infty \left((1-\epsilon)\left(1 + C_{1,2}^{\mathrm{HR}}\right) - C_{1,2}^{\mathrm{HR}} \right) \frac{|u_n(r)|^2}{r^4}\, dr.
\end{aligned}$$

Fixing $\epsilon = 1/(1 + C_{1,2}^{\mathrm{HR}})$ we get that

$$(1-\epsilon)(1 + C_{1,2}^{\mathrm{HR}}) - C_{1,2}^{\mathrm{HR}} = 0$$

and thus

$$h_0[u] \geq g_0^{(1)}\left[P_1 \mathscr{U} u\right] + \epsilon g_0^{(2)}\left[P_2 \mathscr{U} u\right],$$

for any $u \in C_0^\infty\left(\mathbb{R}^3 \backslash \{0\}\right)$. Since $C_0^\infty\left(\mathbb{R}^3 \backslash \{0\}\right)$ is initial domain of h_0 and since the initial domains of $g_0^{(1)}$ and $g_0^{(2)}$ are $P_1 \mathscr{U} C_0^\infty\left(\mathbb{R}^3 \backslash \{0\}\right)$ and $P_2 \mathscr{U} C_0^\infty\left(\mathbb{R}^3 \backslash \{0\}\right)$, respectively, it follows that

$$\mathscr{U} H_0 \mathscr{U}^{-1} \geq G_0^{(1)} \oplus \epsilon G_0^{(2)}.$$

Choose $V \in C_0^\infty\left(\mathbb{R}^3 \backslash \{0\}\right)$ and let $W = \mathscr{U} V \mathscr{U}^{-1}$, $V^{(1)} = P_1 W P_1$ and $V^{(2)} = P_2 W P_2$. Since W is bounded and non-negative, it follows for any $f \in \mathscr{H}$ that

$$2\,\mathrm{Re}(P_1 W P_2 f, f) \leq 2 \left\| W^{1/2} P_2 f \right\| \left\| W^{1/2} P_1 f \right\| \leq (P_1 W P_1 f, f) + (P_2 W P_2 f, f).$$

Hence

$$P_1 W P_2 + P_2 W P_1 \leq V^{(1)} + V^{(2)},$$

and therefore

$$\begin{aligned}
\mathscr{U} H_0 \mathscr{U}^{-1} - W &\geq \left(G_0^{(1)} \oplus \epsilon\, G_0^{(2)} \right) - W \\
&\geq \left(G_0^{(1)} - 2V^{(1)} \right) \oplus \epsilon \left(G_0^{(2)} - \frac{2}{\epsilon} V^{(2)} \right).
\end{aligned}$$

The result now follows from Lemmas 4.5 and 4.7. □

5 Proof of the Main Results

We are now in a position to prove Theorems 2.1, 2.2 and 2.3. This will be accomplished by approximating general potentials with smooth, compactly supported ones, as in the abstract lemma below, and then combine this with Corollary 4.2 and Propositions 4.3 and 4.8.

Lemma 5.1. *Let Ω be an open subset of $\mathbb{R}^d$. Suppose that H_0 is a non-negative symmetric operator in $L^2(\Omega)$, defined on $C_0^\infty(\Omega)$. Let $\hat{H}_0$ be the Friedrich extension of H_0, and choose $\gamma_1 \geq 0$, $\gamma_2 \geq 1$ and $\nu \geq 0$. Suppose that there is a constant $C > 0$, such that for any non-negative $V \in C_0^\infty(\Omega)$, the negative spectrum of $\hat{H}_0 - V$ is discrete, and*

$$\operatorname{tr}(\hat{H}_0 - V)_-^{\gamma_1} \leq C \int_\Omega V(x)^{\gamma_2} |x|^\nu \, dx.$$

Then, for any non-negative potential $\hat{V}$ such that $\hat{V}(x)^{\gamma_2} x^\nu$ is integrable on Ω, the quadratic form

$$h_{\hat{V}}[u] = (H_0 u, u) - \int_\Omega \hat{V}(x) |u(x)|^2 \, dx, \quad u \in C_0^\infty(\Omega)$$

is semi-bounded. Furthermore, if we let $\hat{H}_0 - \hat{V}$ be the self-adjoint operator associated with the closure of the above form, then the negative spectrum of $\hat{H}_0 - \hat{V}$ is discrete and

$$\operatorname{tr}\left(\hat{H}_0 - \hat{V}\right)_-^{\gamma_1} \leq \int_\Omega \hat{V}(x)^{\gamma_2} |x|^\nu \, dx.$$

Proof. Choose a non-negative $\hat{V}$ such that

$$D := \int_\Omega \hat{V}(x)^{\gamma_2} |x|^\nu \, dx < \infty.$$

Also choose a sequence $0 \leq V_1 \leq V_2 \leq \cdots \leq \hat{V}$ such that each V_n belongs to $C_0^\infty(\Omega)$ and such that for almost any $x \in \Omega$,

$$V_n(x) \to \hat{V}(x), \quad \text{as } n \to \infty.$$

By assumption, we know that the negative spectrum of $\hat{H}_0 - V_n$ is discrete and

$$\operatorname{tr}(\hat{H}_0 - V_n)_-^{\gamma_1} \leq C \int_\Omega V_n(x)^{\gamma_2} |x|^\nu \, dx \leq CD. \tag{5.1}$$

Hence in particular,

$$\inf \sigma\left(\hat{H}_0 - V_n\right) \geq -(CD)^{1/\gamma_1}.$$

Note that for given $u \in C_0^\infty(\Omega)$, monotone convergence shows that

$$h_{\hat{V}}[u] = \lim_{n \to \infty} \left((H_0 u, u) - \int_\Omega V_n |u|^2 \, dx \right)$$

and therefore $h_{\hat{V}}$ is lower semi-bounded by $-(CD)^{1/\gamma_1}$ on $C_0^\infty(\Omega)$. Recall that the operator $\hat{H}_0 - \hat{V}$ is defined as the self-adjoint operator associated with the closure of $h_{\hat{V}}$.

Let E and E_n be the spectral measures corresponding to $\hat{H}_0 - \hat{V}$ and $\hat{H}_0 - V_n$, respectively. For $\lambda > 0$, let

$$N(\lambda) = \operatorname{rank} E(-\infty, -\lambda) \quad \text{and} \quad N(\lambda, n) = \operatorname{rank} E_n(-\infty, -\lambda).$$

For fixed $\lambda > 0$, Glazman's lemma shows that

$$N(\lambda, 1) \leq N(\lambda, 2) \leq \cdots \leq N(\lambda),$$

and since

$$N(\lambda, n) \cdot \lambda^{\gamma_1} \leq \operatorname{tr}(\hat{H}_0 - V_n)_-^{\gamma_1} \leq CD,$$

it must be that

$$N(\lambda, \infty) := \lim_{n\to\infty} N(\lambda, n) < \infty.$$

Since $N(\lambda, n)$ only assumes integer values, it follows that there is an integer $m = m(\lambda) \geq 1$ such that

$$N(\lambda, \infty) = N(\lambda, n), \quad n \geq m.$$

Let us prove that for any $\lambda > 0$,

$$N(\lambda) = N(\lambda, \infty). \tag{5.2}$$

Fix $\lambda > 0$ and let $N = N(\lambda, \infty)$. Assume that $N(\lambda) > N$. Then in fact it is possible to find a $\delta > 0$ such that $N(\lambda + \delta) > N$. Again using Glazman's lemma we find a linear set $F \subset C_0^\infty(\Omega)$ with $\dim F = N + 1$ such that

$$h_{\hat{V}}[f] < -(\lambda + \delta)\|f\|^2 \tag{5.3}$$

for any $f \in F$ with $f \neq 0$. Let $\{f_1, f_2, \ldots, f_{N+1}\}$ be an orthonormal basis in F. Use monotone convergence to fix $n \geq 1$ such that

$$(N+1)\int_\Omega (\hat{V} - V_n)|f_k|^2\, dx < \delta, \quad k = 1, 2, \ldots, N+1. \tag{5.4}$$

Now, since $\dim F > N(\lambda, n)$ there must be scalars $c_1, c_2, \ldots, c_{N+1}$, not all zero, such that $g := c_1 f_1 + c_2 f_2 + \cdots + c_{N+1} f_{N+1}$ satisfies

$$(H_0 g, g) - \int_\Omega V_n |g|^2\, dx \geq -\lambda \|g\|^2.$$

Note that $|c_1|^2 + |c_2|^2 + \cdots + |c_{N+1}|^2 = \|g\|^2$. Now, by (5.4)

$$\begin{aligned} h_{\hat{V}}[g] &= (H_0 g, g) - \int_\Omega V_n |g|^2\, dx - \int_\Omega (\hat{V} - V_n)|g|^2 \\ &\geq -\lambda\|g\|^2 - (N+1)\sum_{k=1}^{N+1} |c_k|^2 \int_\Omega (\hat{V} - V_n)|f_k|^2\, dx \\ &\geq -(\lambda + \delta)\|g\|^2. \end{aligned}$$

This contradicts (5.3), and therefore (5.2) holds.

By (5.2), the negative spectrum of $\hat{H} - \hat{V}$ is discrete. Denote by λ_j and $\lambda_{j,n}$, where $j = 1, 2, 3, \ldots$, the negative eigenvalues of $\hat{H}_0 - \hat{V}$ and $\hat{H}_0 - V_n$, respectively, ordered such that

$$\lambda_1 \leq \lambda_2 \leq \cdots \quad \text{and} \quad \lambda_{1,n} \leq \lambda_{2,n} \leq \cdots.$$

For convenience, we always consider infinite sequences λ_j and $\lambda_{j,n}$. If there should only be finitely many, say k, negative eigenvalues for the operator $\hat{H}_0 - \hat{V}$, or $\hat{H}_0 - V_n$, we let $\lambda_j = 0$, or $\lambda_{j,n} = 0$, for $j > k$. Note that

$$\lambda_{j,1} \geq \lambda_{j,2} \geq \cdots \geq \lambda_j.$$

Using (5.2) again, we see that $\lambda_{j,n} \to \lambda_j$ as $n \to \infty$, and the result now follows from (5.1). $\square$

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